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1991 J. Phys. A: Math. Gen. 24 L169

(http://iopscience.iop.org/0305-4470/24/4/002)

J. Phys. A: Math. Gen. 24 (1991) L169–L174. Printed in the UK

LETTER TO THE EDITOR

Comment on the differential calculus on quantum planes

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Received 5 December 1990

Abstract. As a good introduction to and example of the differential calculus on quantum vector spaces pioneered by Wess and Zumino we consider two-parameter quantum deformation of the plane and its calculus. The main structures are derived anew without an R matrix and their generalizations to multiparameter deformation and to quantum superspace are briefly discussed in an arbitrary number of dimensions.

In an important conceptual advance, Wess and Zumino [1] have shown recently that consistent quantum deformation of the differential calculus is provided by an R matrix which can be any solution of the quantum Yang–Baxter equation. Schirrmacher [2], who applied their formalism to quantum planes, was led to multiparameter deformation of the differential calculus thereon. For this purpose it was sufficient to insert the multiparametric R matrix [2–4] into the formulae of Wess and Zumino which are given in terms of the R matrix. The full generality of their approach is very appealing. Nonetheless it seems timely to remark that their explicit structures can all be recovered without the R matrix with the help of a restricted ansatz. As a concrete example of this alternative approach we consider the two-dimensional quantum vector space. It is straightforward to generalize our considerations to an arbitrary-dimensional hyperplane spanned by ordinary [5] or by supernumerary [6] coordinates.

Let the quantum plane coordinates be denoted x and y and their differentials

$$X = \mathrm{d}x \qquad Y = \mathrm{d}y. \tag{1}$$

The basic commutation relations are

$$xy - qyx = 0 \tag{2}$$

where q is a complex deformation parameter. To develop the calculus on the plane defined in terms of basic variables satisfying (2), we shall give the following ansatz for the commutation relations between the variables and their differentials. Let us assume (see also the closing remark in the paragraph before equation (17))

$$\begin{aligned} xX &= r_1 Xx \qquad xY = F_{11}Xy + F_{12}Yx \\ yX &= F_{21}Xy + F_{22}Yx \qquad yY = r_2 Yy. \end{aligned} \tag{3}$$

Following Kobyzev and Manin [5] the quantum deformed GL(2) acts as a linear transformation on the quantum plane which preserves the basic relations (2) and their 'dual'. Extending this property of covariance (under the action of quantum GL(2)), from the plane to its calculus, it will be assumed that the quantized group structure implies and is implied by invariance of the 'intermediary' relations (3) under linear

0305-4470/91/040169+06$03.50

transformations of the quantum plane. For the first or the fourth relations in the set (3) to be preserved at all plane endomorphisms we require at least

$$r_1 = r_2 = r \qquad \text{(say).} \tag{4}$$

The coefficients denoted F in (3) can be related to q and r by the consistency of calculus. Let us apply the exterior derivative d operation to the basic relations (2). If by this the left-hand side is still to vanish by virtue of (3), it is required that

$$F_{11} = qF_{21} - 1 \qquad F_{22} = 1/q \qquad F_{12} = 1. \tag{5}$$

In arriving at this result we used the chain rule obeyed by the d operator

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f\,\mathrm{d}g. \tag{6}$$

Furthermore the linear relations (3) should also allow the differentials to be pulled through the quadratic relations (2) from one side to the other. This further demands

$$F_{11}F_{22} = F_{22}(F_{12} - qr) = F_{11}(r - qF_{21}) = 0. \tag{7}$$

In fact (5) and (7) are in disguise (the restricted form of) the 'linear' and 'quadratic' consistency conditions discussed in full generality by Wess and Zumino. Equation (7) admits two solutions. If we choose F_{22} to vanish, we are led to the following two-parameter deformed intermediary relations represented in terms of q and r

$$\begin{aligned} &xX = rXx \qquad && xY = qYx + (r-1)Xy \\ &yX = \frac{r}{q}Xy \qquad && yY = rYy. \end{aligned} \tag{8}$$

Applying the d operation on (8) and utilizing the nilpotency of d it is easy to see that $(p = r/q)$

$$XY = -(1/p)YX \qquad X^2 = Y^2 = 0. \tag{9}$$

Hence the differentials of the q plane span the $1/p$ exterior plane.

It is known [7] that the relations (2) and (9) (defining the q plane and the $1/p$ exterior plane respectively) are preserved under

$$\begin{aligned} &x \to T^1_1 x + T^1_2 y \qquad && y \to T^2_1 x + T^2_2 y \\ &x \to T^1_1 X + T^1_2 Y \qquad && Y \to T^2_1 X + T^2_2 Y \end{aligned} \tag{10}$$

provided the matrix elements (assumed to commute with x, y, X and Y) are subject to

$$\begin{aligned} &T^1_1 T^1_2 = pT^1_2 T^1_1 \qquad && T^2_1 T^2_2 = pT^2_2 T^2_1 \\ &T^1_1 T^2_1 = qT^2_1 T^1_1 \qquad && T^1_2 T^2_2 = qT^2_2 T^1_2 \\ &T^1_2 T^2_1 = \frac{q}{p} T^2_1 T^1_2 \qquad && T^1_1 T^2_2 = T^2_2 T^1_1 + \left(q - \frac{1}{p}\right) T^2_1 T^1_2. \end{aligned} \tag{11}$$

This is just the condition for the transformation coefficients to belong to the $\mathrm{GL}_{r,q}(2)$ matrix. In the same sense the relations (8) are preserved under the action of the quantum matrix (T^i_j) defined by (11).

To complete the differential geometric scheme let us introduce derivatives of the quantum plane in the standard way through

$$d = Xd_x + Yd_y. \tag{12}$$

Here we have denoted

$$d_x = \partial/\partial x \qquad d_y = \partial/\partial y. \tag{13}$$

From $d^2 = 0$ it is easy to infer the commutation relations among the derivatives

$$d_x d_y = \frac{1}{p} d_y d_x. \tag{14}$$

Applying the chain rule (6) to basic variables we arrive at the commutation relations between derivatives and variables

$$\begin{aligned} d_x x &= 1 + rxd_x + (r-1)yd_y \qquad d_x y = pyd_x \\ d_y x &= qxd_y \qquad d_y y = 1 + ryd_y. \end{aligned} \tag{15}$$

Finally the rule to commute derivatives through differentials must be

$$\begin{aligned} d_x X &= \frac{1}{r} X d_x \qquad d_x Y = \frac{1}{q} Y d_x \qquad d_y X = \frac{1}{p} X d_y \\ d_y Y &= \frac{1}{r} Y d_y + \left(\frac{1}{r} - 1\right) X d_x. \end{aligned} \tag{16}$$

This is so in order that unmatched terms linear in differentials are not generated (or else the scheme becomes inconsistent) when basic variables are pulled through to one side of (16) by means of the inverse of relations (8).

The relations (2), (8), (9), (14), (15) and (16), comprising the two-parameter differential geometric scheme, are a generalization of the corresponding relations in the one-parameter scheme considered by Wess and Zumino (see [1] section 4) as the simplest example of their general formalism. The choice $p = q$ made in [1] is rendered rather special in view of the existence of two-parameter deformations. One can even have $q = 1$ but $p \neq 1$, or $q \neq 1$ but $p = 1$, or also $p = 1/q$, so that several of the commutation relations are left undeformed. The two-parameter calculus is covariant with respect to $GL_{r,q}(2)$. All our commutation relations defining the scheme are preserved under linear transformations of the plane (10) subject to the commutation structure in (11). This is easily understood with the help of the known form of the two-parameter R matrix [7] which brings our explicit commutation relations in the form given by Wess and Zumino. In this way it is also realized in retrospect that the ansatz (3) we made in the beginning is compatible with such assumptions on the R matrix as lead Schirrmacher *et al* [7] to two-parameter deformation by solving the quantum Yang–Baxter equation in two dimensions.

Our procedure readily generalizes for n dimensions. We need to make the following substitutions

$$\begin{aligned} &x \to x^i \qquad y \to x^j \qquad X \to X^i \qquad Y \to X^j \\ &q \to q_{ij} \qquad p \to p_{ij} \qquad d_x \to d_i \qquad d_y \to d_j \end{aligned}$$

where the indices $i, j = 1, 2 \ldots n$ subject to $j > i$. The basic relations are

$$x^i x^j = q_{ij} x^j x^i. \tag{17}$$

The associated exterior space is defined by

$$X^i X^j = -\frac{1}{p_{ij}} X^j X^i \qquad (X^i)^2 = 0. \tag{18}$$

Covariance demands that ($r \neq -1$)

$$q_{ij}p_{ij} = r. \tag{19}$$

Hence we have a total of $n(n-1)/2+1$ independent parameters. Our ansatz leads us to the following commutation structure (an identical result is obtained in [2] using the R matrix formalism)

$$\begin{aligned} x^iX^i &= rX^ix^i \\ x^iX^j &= q_{ij}X^jx^i + (r-1)X^ix^j \\ x^jX^i &= p_{ij}X^ix^j. \end{aligned} \tag{20}$$

The commutation relations of the non-commuting derivatives can also be given, with themselves

$$d_id_j = \frac{1}{p_{ij}} d_jd_i \tag{21}$$

with the variables

$$\begin{aligned} d_ix^i &= 1 + x^id_i + (r-1)\sum_{k=i+1}^{n} x^kd_k \\ d_ix^j &= p_{ij}x^jd_i \\ d_jx^i &= q_{ij}x^id_j \end{aligned} \tag{22}$$

and finally with the differentials

$$\begin{aligned} d_iX^i &= \frac{1}{r} X^id_i + \left(\frac{1}{r} - 1\right)\sum_{k=1}^{i-1} X^kd_k \\ d_iX^j &= \frac{1}{d_{ij}} X^id_i \\ d_jX^i &= \frac{1}{p_{ij}} X^id_j. \end{aligned} \tag{23}$$

The entire multiparametric scheme thus developed is covariant with respect to $GL_{r,q_{ij}}(n)$ acting as linear transformations on the hyperplane. The relations (17)–(23), which can be introduced in the elegant form given by Wess and Zumino with the help of the multiparametric R matrix of Schirrmacher [2], Sudbery [3] and Demidov *et al* [4], are all preserved under the transformations (for fixed values of i, j, k, l such that $i<j$ and $k<l$)

$$\begin{aligned} x^i &\to T^i_kx^k + T^i_lx^l \qquad & x^j &\to T^j_kx^k + T^j_lx^l \\ X^i &\to T^i_kX^k + T^i_lX^l \qquad & X^j &\to T^j_kX^k + T^j_lX^l \end{aligned} \tag{24}$$

with the transformation coefficients (assumed to commute with the variables and their differentials) enjoying the commutation structure [2]

$$\begin{aligned} T^i_kT^i_l &= p_{kl}T^i_lT^i_k \qquad & T^j_kT^j_l &= p_{kl}T^j_lT^j_k \\ T^i_kT^j_k &= q_{ij}T^j_kT^i_k \qquad & T^i_lT^j_l &= q_{ij}T^j_lT^i_l \\ T^i_lT^j_k &= \frac{q_{kl}}{p_{ij}} T^j_kT^i_l & & \\ T^i_kT^j_l &= \frac{q_{ij}}{q_{kl}} T^j_lT^i_k + \left(q_{ij} - \frac{1}{p_{ij}}\right) T^j_kT^i_l. & & \end{aligned} \tag{25}$$

Equations (25) may be compared with (11). It is evident that the last relation in the set (25) is further deformed as compared with the corresponding commutation relation in (11).

Analogous considerations apply also to the quantum superspace. We shall state here only the results for the differential geometric scheme formulated on the $(n+m)$-dimensional quantum superspace led to by the requirement of consistency and covariance with respect to $\mathrm{GL}_{r;q_{IJ}}(n/m)$. Here q_{IJ} are $(n+m)(n+m-1)/2$ complex deformation parameters, and in the following I and J are ordered so that $I<J$ and $I, J=1, 2 \ldots n+m$. The indices I, J are supernummerary with

$$\hat{I}, \hat{J}=0 \qquad I, J=1, 2, \ldots n$$
$$\hat{I}, \hat{J}=1 \qquad I, J=n+1, n+2 \ldots n+m.$$

The basic algebra is

$$\begin{aligned} x^I x^J &= (-1)^{\hat{I}\hat{J}} \frac{1}{q_{IJ}} x^J x^I \\ (x^{n+1})^2 &= (x^{n+2})^2 = \ldots = (x^{n+m})^2 = 0. \end{aligned} \tag{26}$$

The associated exterior space identified with the differentials of the basic variables is

$$\begin{aligned} X^I X^J &= (-1)^{(\hat{I}+1)(\hat{J}+1)} \frac{1}{p_{IJ}} X^J X^I \\ (X^1)^2 &= (X^2)^2 = \ldots = (X^n)^2 = 0 \end{aligned} \tag{27}$$

with $p_{IJ}=r/q_{IJ}$.

The intermediary relations obtained with the help of our ansatz are

$$\begin{aligned} x^I X^I &= (r)^{1-\hat{I}} x^I x^I \\ x^I X^J &= q_{IJ} X^J x^I + (-1)^{\hat{I}}(r-1) X^I x^J \\ x^J X^I &= (-1)^{\hat{J}(\hat{I}+1)} p_{IJ} X^I x^J. \end{aligned} \tag{28}$$

From $d^2=0$ we infer the (anti-) commutation relations between derivatives of the superspace

$$d_I d_J = (-1)^{\hat{I}\hat{J}} \frac{1}{p_{IJ}} d_J d_I. \tag{29}$$

From the graded chain rule applied to the basic variables we arrive at the (anti-) commutation relations between derivatives and variables

$$\begin{aligned} d_I x^I &= 1 + (-1)^{\hat{I}} x^I d_I + (r-1) \sum_{K=I+1}^{n+m} x^K d_K \\ d_I x^J &= (-1)^{\hat{I}\hat{J}} q_{IJ} x^J d_I \\ d_J x^I &= (-1)^{\hat{I}\hat{J}} q_{IJ} x^I d_J. \end{aligned} \tag{30}$$

Finally

$$\begin{aligned} d_I X^I &= \left(\frac{1}{r}\right)^{1-\hat{I}} X^I d_I + \left(\frac{1}{r}-1\right) \sum_{K=1}^{I-1} (-1)^{\hat{K}} X^K d_K \\ d_I X^J &= \frac{1}{d_{IJ}} X^J d_I \\ d_J X^I &= (-1)^{\hat{J}(\hat{I}+1)} \frac{1}{p_{IJ}} X^I d_J. \end{aligned} \tag{31}$$

By virtue of (28) and (31) all the differentials can be consistently pulled through (30) from one side to the other. The invariance of the scheme defined by (26)–(31) under linear transformations of the quantum superspace leads to the (anti-) commutation structure of the quantum supergroup $GL_{r,q_{ij}}(n/m)$ [6].

This work was supported by CSIR, New Delhi, India.

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